# FURTHER MATHEMATICS

**Paper 9231/11**
**Paper 11**

**Key messages:**

- Candidates should make sure that they learn formulae and methods thoroughly so that they can apply them effectively to solving problems
- Candidates should show all working fully, taking particular care to identify the general case in an induction proof
- Candidates should follow instructions on the paper carefully, making sure that they give answers in the form required, use prescribed methods and do not miss any question parts.

**General Comments**

Many candidates completed this paper to a very high standard, showing excellent communication skills in their solutions. A small number of candidates were not able to demonstrate their knowledge in their solutions but most completed the paper fully, and some had time to attempt both options for **Question 11**. Algebraic handling was generally very good, and most candidates were able to sustain their accuracy over longer manipulations. Matrix techniques were well demonstrated, and relatively few errors were seen in handling calculus on most scripts.

**Question 1**

This question was well done by the majority of candidates, who were able to split the denominator into partial fractions correctly. A small number did not follow the prescribed method of differences. Once they had found the sum of *n* terms, candidates were able to find the sum to infinity.

*Answer*: $\frac{1}{2}\left(1-\frac{1}{2n+1}\right), \frac{1}{2}$

**Question 2**

Most candidates found *q*, but a number could not recall a formula for the sum of cubes and spent some time trying to reconstruct it, often with inaccuracies. The more direct method of adding three cubic equations together usually led to a correct answer for the product of roots.

*Answer*: $x^3 - 3x^2 + 4x + 7 = 0$

**Question 3**

This question was very well done either by the standard method of constructing two matrices, one with the eigenvalues (**D**) and one with the eigenvectors (**P**), and calculating $\mathbf{P}^{-1}$ and the product $\mathbf{PDP}^{-1}$**.** Others equally effectively used the products of the matrix **A** with each eigenvector in turn and equated these to the product of the eigenvalue and eigenvector in each case.

*Answer*: $\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

**Question 4**

Most candidates were able to manipulate the factorial expressions to prove the given result. In general, candidates were able to check the hypothesis for $n$=1 and state the assumption for $n$=$k$, although sometimes did not use the correct terminology. Very few included a general term in their inductive proof, and even fewer applied the result of the first part of the question to prove the inductive step.

**Question 5**

The first part of this question was very well done, with candidates showing the steps clearly in the row reduction process. Some were unsure what constituted the basis for the range space, but most were able to find the two vectors for the basis of the null space, though some forgot to multiply all the elements in the vector when trying to make the elements integers.

*Answers*: **(i)** r(**A**) = 4 – 2 = 2 **(ii)** $\left\{\begin{pmatrix}1\\2\\3\\6\end{pmatrix}, \begin{pmatrix}3\\8\\13\\24\end{pmatrix}\right\}$ **(iii)** $\left\{\begin{pmatrix}-29\\5\\0\\2\end{pmatrix}, \begin{pmatrix}-19\\3\\2\\0\end{pmatrix}\right\}$

**Question 6**

Candidates were almost all confident in solving this differential equation, though a small number forgot the correct form of the particular integral in this case. Most noticed and used the correct variables throughout the question, though a minority reverted to $x$ as the independent as well as dependent variable in their complementary function. They were able to identify the approximate solution for large $t$ from their knowledge of exponential functions.

*Answer*: GS: $x = Ae^{-2t} + Be^{-5t} + 3\sin 2t - 7\cos 2t$, Approximate solution: $x \approx 3\sin 2t - 7\cos 2t$.

**Question 7**

A number of candidates found the mean value of $y$, instead of $\frac{dy}{dx}$ in the first part of this question. Most candidates remembered the formulae for the coordinates of the centroid, and were able to calculate the various component integrations correctly, although some did not complete the question fully by calculating the numerical values as requested.

*Answer*: **(i)** –0.491, **(ii)** $\bar{x} = 0.463$, $\bar{y} = 0.255$

**Question 8**

Many candidates completed this question effectively, applying implicit differentiation accurately to find both first and second derivatives. Several found the coordinates of the stationary points correctly, but switched the $x$ and $y$ coordinates round when determining whether the points were maximum or minimum.

*Answer*: Maximum at (4, –2) and minimum at (–4, 2)

**Question 9**

Most candidates were able to integrate by parts to find the required integral, and the majority took the simpler route to finding the given reduction formula. Many of those who split the integrand into $-x^{n-1}\, x\sin x$ eventually reached the correct answer too. It was important to identify the separate steps in the final integration by substitution to avoid sign errors.

*Answer*: $\int_0^{\frac{\pi}{2}} x\sin x\, dx = 1$, $\frac{3}{4}\pi^2 - 6$

**Question 10**

The first part of this question was well done. Most candidates were able to multiply out the two brackets, and some noticed groupings that simplified the process. Those who used trigonometric substitutions before multiplying the two brackets were not always able to reduce the expression as required, though some used the formula sheet sensibly to complete the process correctly. The final integral was successfully completed with only occasional slips, often with the constant term.

*Answer*: $\frac{3\pi - 4}{192}$ or 0.0283

**Question 11 EITHER**

This was the less popular choice, but those who chose it mainly completed the first part well, despite some complicated algebra. Most chose to use the scalar product, though a few found the direction vector of *PQ* and matched it to their expression for *PQ*, usually remembering to include a scalar multiplier. Although candidates were able to identify a point on the line and find one vector in the required plane, many were unable to identify a second vector in the plane in the second part. The final part of the question proved challenging, with only the stronger candidates being able to find a point on the line and its direction too.

*Answer*: $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$,

$\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

**Question 11 OR**

Candidates were able to attempt the first three parts of this question with competence, most remembering the formulae for arc length and surface area in parametric form, as well as proving the initial (and vital) identity. Some omitted to find the cartesian equation as required before going on to show the polar equation, and others did not find the domain correctly. Strong candidates were able to use their previous answers to find the required region, recognising the integrand as a function of $\tan\theta$ preceded by its derivative. Others went through a formal substitution, but a number of candidates substituted for $\tan\theta$ and could not complete the integration.

*Answer*: **(i)** $\frac{2}{\sqrt{3}}$ **(ii)** $\frac{\pi}{3}$, $x = 1 - 3\frac{y^2}{x^2}$; $0 \le \theta \le \frac{\pi}{6}$, $\frac{4}{45}\sqrt{3}$

# FURTHER MATHEMATICS

**Paper 9231/12**
**Paper 12**

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**Paper 9231/13**
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# FURTHER MATHEMATICS

**Paper 9231/21**
**Paper 21**

## Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers.

## General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though most of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with parts of **Questions 3, 4, 8 and 9** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, the directions of motion of particles or what forces are acting and also their directions as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 6 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen for example when taking the difference of means or estimating variances.

**Comments on specific questions**

**Question 1**

Almost all candidates found the value of $\omega^2$, here 2, from the standard SHM formula $v^2 = \omega^2 (a^2 - x^2)$ with $v$ = 4, $a$ = 3 and $x$ = 1. This enables the magnitude of the maximum acceleration to be found from $\omega^2 a$, though a few candidates found the maximum velocity instead. While the number of oscillations made by $P$ in one minute follows immediately from $60/T$ where the period $T = 2\pi/\omega$, candidates should note that the number of complete oscillations is the integer part of this result. The required time $t_{AC}$ from $A$ to $C$ may be found in a variety of ways, but in all cases it is important to consider whether the expression $a \sin \omega t$ or $a \cos \omega t$ is the appropriate one for the distance being found. The most direct method is $3 \cos \omega t_{AC} = -1$, but another popular method was to find the time from $M$ to $C$ using $3 \sin \omega t_{MC} = 1$ and then add the time $\frac{1}{4}T$ from $A$ to $M$.

*Answers*: **(i)** 6 m s$^{-2}$; **(ii)** 13; **(iii)** 1·35 s.

**Question 2**

The familiar advice to first read the question is appropriate here, since the instruction "By considering the collision at $E$" in part **(i)** did not dissuade many candidates from first considering the collision at $D$ instead, which is of no help in this part. The given result may be verified by equating the sum of the squares of the two components $v \cos 45°$ and $\frac{3}{4}v \sin 45°$ to $(\frac{1}{4}u)^2$, or equivalently by first finding $\tan \beta = \frac{3}{4}$ where $\beta$ is the angle between the final direction of motion and the second wall. Turning to the collision at $D$, equating the component of $P$'s speed parallel to the first wall before and after the collision yields the value of $\alpha$, while application of the restitution principle to the component of speed normal to the wall then gives the value of $e$. Since candidates are considering speeds rather than velocities, they should not apply minus signs to any of the components.

*Answers*: **(ii)** 85·8°, 0·274.

**Question 3**

Verifying the given value of $\sin \theta$ from the right-angled triangle with hypotenuse $CD$ was invariably successful. Most candidates then used vertical resolution of forces to find the reaction $R_A$ at $A$ in the form $(k + 2)W$, although this is in fact unnecessary in this part if moments are taken about the centre of the rod to give $7F_A = 4R_A$. Most candidates chose instead to take moments for the system about some other point, of which there are several to choose from, in order to find the friction $F_A$ at $A$ in the form $(4/7)(k + 2)W$. The numerical value of $\mu$ then follows from $F_A / R_A$. Candidates should not take moments for either sphere about the point $P$ or $Q$ where the rod is attached to the sphere, since the rigid nature of the attachment means that the sum of moments is not necessarily zero. The given value of $k$ may be readily verified in the second part by equating $F_A^2 + R_A^2$ to $65W^2$, provided of course that the correct value of $\mu$ has been found.

*Answers*: **(ii)** 4/7.

**Question 4**

Using conservation of energy to find the speed of the particle $P$ just before the impact, and then conservation of momentum to verify it's given speed immediately after the impact rarely presented problems, except where candidates attempted to combine these two processes into a single step. Having used the fact that the mass of the combined particle is $(\lambda + 1)m$ in the first part, candidates should continue to use this mass instead of $m$ in the second part, but not all did so. As it happens, the required value of $\lambda$ is independent of which mass is used, provided the same mass is used consistently throughout part **(ii)**, but inconsistency produces an incorrect result. The method of deriving $\lambda$ is to combine conservation of energy with the application of Newton's second law of motion to the particle in a radial direction when the string becomes slack in order to eliminate the speed at this point. The resulting quadratic in $\lambda$ is then solved for the positive root. In the final part, Newton's second law of motion is applied again in a radial direction to produce the tensions $28mg/3$ and $20mg/3$ in the string immediately before and after the collision respectively, and hence the change in tension. Once again it is essential to use the appropriate particle mass in each equation.

*Answers*: **(ii)** ⅔; **(iii)** $8mg/3$.

**Question 5**

Almost all candidates were aware that the required value of the parameter $a$ is the reciprocal of the mean value of $X$, and then evaluated $1 - e^{-15000a}$ to find the required probability. Care must be taken in the final part to equate the given probability 0·75 to 1 – F($d$) rather than F($d$).

*Answers*: **(i)** 0·0001; **(ii)** 0·777; **(iii)** 2880.

**Question 6**

This question was usually answered well. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. Using an estimated population variance of 0·112, the $t$-value is found to be 1·64. Since it is a one-tailed test, comparison with the tabulated value of 1·895 leads to acceptance of the null hypothesis, thus concluding that the population mean of $X$ is not less than 6·44.

**Question 7**

Most candidates appreciated that the distribution function F of $X$ over $2 \le x \le 4$ is found by integrating f(x), but not all considered also the intervals $x < 2$ and $x > 4$. The next step is to determine the distribution function G of $Y$ over the interval corresponding to $2 \leqslant x \leqslant 4$, which is here $F(y^{1/3}) = (y^{2/3} - 4)/12$ over $8 \leqslant y \leqslant 64$. Differentiation then gives the required non-zero expression for g($y$), and for completeness candidates should include the relevant interval. Much as in Question **(5)(iii),** care must be taken in the final part here to equate the given value 7/12 to 1 – G($k$) rather than G($k$).

*Answers*: **(i)** 0 ($x < 2$), $(x^2 - 4)/12$ ($2 \leqslant x \leqslant 4$), 1 ($x > 4$); **(ii)** $1/(18y^{1/3})$ ($8 \leqslant y \leqslant 64$), 0 otherwise;
**(iii)** 27.

**Question 8**

It is essential that candidates can distinguish between a pooled estimate for a common population variance, as explicitly required here, and an estimate of a combined variance. Apart from this occasional confusion, most candidates knew how to find a confidence interval for the difference in the population means, though a few mistakenly interpreted the question as requiring a test of the means. While most used the tabular $z$-value 1·96, some preferred the closest $t$-value in the *Statistical Tables* to that required here, namely 1·98. The latter choice gives a half-interval width of 7·3 rather than 7·2.

*Answers*: **(i)** 368; **(ii)** $16 \pm 7{\cdot}2$ .

**Question 9**

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*Answers*: **(i)** 3, 5; **(ii)** 0·949 .

# FURTHER MATHEMATICS

**Paper 9231/22**
**Paper 22**

## Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

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Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers.

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Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though most of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with parts of **Questions 3, 4, 8 and 9** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, the directions of motion of particles or what forces are acting and also their directions as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 6 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen for example when taking the difference of means or estimating variances.

**<u>Comments on specific questions</u>**

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Almost all candidates found the value of $\omega^2$, here 2, from the standard SHM formula $v^2 = \omega^2 (a^2 - x^2)$ with $v$ = 4, $a$ = 3 and $x$ = 1. This enables the magnitude of the maximum acceleration to be found from $\omega^2 a$, though a few candidates found the maximum velocity instead. While the number of oscillations made by $P$ in one minute follows immediately from $60/T$ where the period $T = 2\pi/\omega$, candidates should note that the number of complete oscillations is the integer part of this result. The required time $t_{AC}$ from $A$ to $C$ may be found in a variety of ways, but in all cases it is important to consider whether the expression $a \sin \omega t$ or $a \cos \omega t$ is the appropriate one for the distance being found. The most direct method is $3 \cos \omega t_{AC} = -1$, but another popular method was to find the time from $M$ to $C$ using $3 \sin \omega t_{MC} = 1$ and then add the time $\frac{1}{4}T$ from $A$ to $M$.

*Answers*: **(i)** 6 m s$^{-2}$;  **(ii)** 13;  **(iii)** 1·35 s.

**Question 2**

The familiar advice to first read the question is appropriate here, since the instruction "By considering the collision at $E$" in part **(i)** did not dissuade many candidates from first considering the collision at $D$ instead, which is of no help in this part. The given result may be verified by equating the sum of the squares of the two components $v \cos 45°$ and $\frac{3}{4}v \sin 45°$ to $(\frac{1}{4}u)^2$, or equivalently by first finding $\tan \beta = \frac{3}{4}$ where $\beta$ is the angle between the final direction of motion and the second wall. Turning to the collision at $D$, equating the component of $P$'s speed parallel to the first wall before and after the collision yields the value of $\alpha$, while application of the restitution principle to the component of speed normal to the wall then gives the value of $e$. Since candidates are considering speeds rather than velocities, they should not apply minus signs to any of the components.

*Answers*: **(ii)** 85·8°, 0·274.

**Question 3**

Verifying the given value of $\sin \theta$ from the right-angled triangle with hypotenuse $CD$ was invariably successful. Most candidates then used vertical resolution of forces to find the reaction $R_A$ at $A$ in the form $(k + 2)W$, although this is in fact unnecessary in this part if moments are taken about the centre of the rod to give $7F_A = 4R_A$. Most candidates chose instead to take moments for the system about some other point, of which there are several to choose from, in order to find the friction $F_A$ at $A$ in the form $(4/7)(k+2)W$. The numerical value of $\mu$ then follows from $F_A / R_A$. Candidates should not take moments for either sphere about the point $P$ or $Q$ where the rod is attached to the sphere, since the rigid nature of the attachment means that the sum of moments is not necessarily zero. The given value of $k$ may be readily verified in the second part by equating $F_A^2 + R_A^2$ to $65W^2$, provided of course that the correct value of $\mu$ has been found.

*Answers*: **(ii)** 4/7.

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Using conservation of energy to find the speed of the particle $P$ just before the impact, and then conservation of momentum to verify it's given speed immediately after the impact rarely presented problems, except where candidates attempted to combine these two processes into a single step. Having used the fact that the mass of the combined particle is $(\lambda + 1)m$ in the first part, candidates should continue to use this mass instead of $m$ in the second part, but not all did so. As it happens, the required value of $\lambda$ is independent of which mass is used, provided the same mass is used consistently throughout part **(ii),** but inconsistency produces an incorrect result. The method of deriving $\lambda$ is to combine conservation of energy with the application of Newton's second law of motion to the particle in a radial direction when the string becomes slack in order to eliminate the speed at this point. The resulting quadratic in $\lambda$ is then solved for the positive root. In the final part, Newton's second law of motion is applied again in a radial direction to produce the tensions $28mg/3$ and $20mg/3$ in the string immediately before and after the collision respectively, and hence the change in tension. Once again it is essential to use the appropriate particle mass in each equation.

*Answers*: **(ii)** ⅔;  **(iii)** $8mg/3$.

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Almost all candidates were aware that the required value of the parameter $a$ is the reciprocal of the mean value of $X$, and then evaluated $1 - e^{-15\,000a}$ to find the required probability. Care must be taken in the final part to equate the given probability 0·75 to 1 – F($d$) rather than F($d$).

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This question was usually answered well. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. Using an estimated population variance of 0·112, the $t$-value is found to be 1·64. Since it is a one-tailed test, comparison with the tabulated value of 1·895 leads to acceptance of the null hypothesis, thus concluding that the population mean of $X$ is not less than 6·44.

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Most candidates appreciated that the distribution function F of $X$ over $2 \le x \le 4$ is found by integrating f(x), but not all considered also the intervals $x < 2$ and $x > 4$. The next step is to determine the distribution function G of $Y$ over the interval corresponding to $2 \leqslant x \leqslant 4$, which is here $F(y^{1/3}) = (y^{2/3} - 4)/12$ over $8 \leqslant y \leqslant 64$. Differentiation then gives the required non-zero expression for g($y$), and for completeness candidates should include the relevant interval. Much as in Question **(5)(iii),** care must be taken in the final part here to equate the given value 7/12 to 1 – G($k$) rather than G($k$).

*Answers*: **(i)** 0 ($x < 2$), $(x^2 - 4)/12$ ($2 \leqslant x \leqslant 4$), 1 ($x > 4$); **(ii)** $1/(18y^{1/3})$ ($8 \leqslant y \leqslant 64$), 0 otherwise;
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# FURTHER MATHEMATICS

**Paper 9231/23**
**Paper 23**

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